Three-Body System with Short-Range Interactions

J. Gegelia[a*]

[a]Department of Physics, Flinders University, Bedford Park, SA 5042 Australia

Within the framework of non-relativistic scalar effective field theory it is shown that the problem of the cutoff dependence of the leading order amplitude for a particle scattering off a two-body bound state can be solved without introducing three-body forces.

Applications of effective field theory (EFT) to problems of nuclear physics have been under intensive investigations during the last few years. A review of recent developments (and references to the relevant papers) can be found in [1].

Generalisation of the EFT program to the three-body problem is not straightforward. In bosonic systems and in some fermionic channels one encounters a non-trivial problem. While each leading order three-body diagram with re-summed two-body interactions is individually finite, the whole amplitude shows sensitivity to the ultraviolet cutoff. In [2] it was argued that the addition of an one-parameter three-body force counter-term at leading order is *necessary and sufficient* to eliminate this cut-off dependence.

The present paper considers the simple case of a non-relativistic scalar particle scattering off a two-body bound state and provides a solution of the above mentioned problem of sensitivity to the ultraviolet cut-off without introducing three-body forces into the leading order Lagrangian.

The Lagrangian of the considered EFT of non-relativistic self-interacting boson ϕ is given by the following expression [3]:

$$\mathcal{L} = \phi^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2M} \right) \phi - \frac{C_0}{2}(\phi^\dagger\phi)^2 - \frac{D_0}{6}(\phi^\dagger\phi)^3 + \ldots, \tag{1}$$

where the ellipsis stands for terms with more derivatives and/or fields. Terms with more derivatives are suppressed at low momentum and terms with more fields do not contribute to the three-body amplitude. For the sake of convenience [4] one can rewrite this theory introducing a dummy field T with the quantum numbers of two bosons (referred to as "dimeron" [3]),

$$\begin{aligned} \mathcal{L} &= \phi^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2M} \right) \phi + \Delta T^\dagger T - \frac{g}{\sqrt{2}}(T^\dagger \phi\phi + \text{h.c.}) \\ &+ hT^\dagger T \phi^\dagger \phi + \ldots \end{aligned} \tag{2}$$

*This work was carried out whilst the author was a recipient of an Overseas Postgraduate Research Scholarship and a Flinders University Research Scholarship at the Flinders University of South Australia.

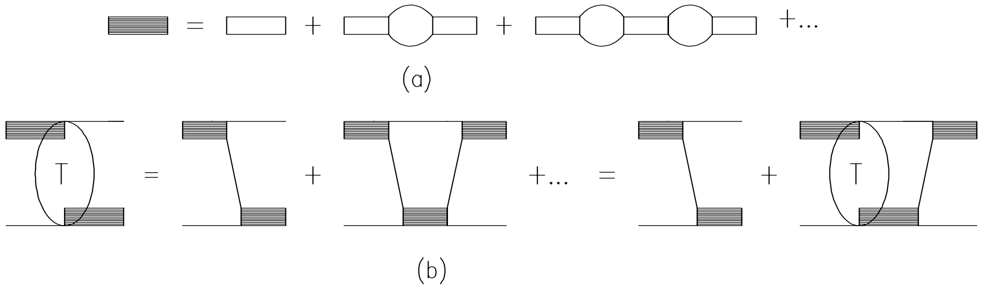

Figure 1. *(a) Dressing of the dimeron. (b) Diagrams contributing to the particle - bound-state scattering.*

Observables depend on the parameters of Eq. (2) only through the combinations $C_0 \equiv g^2/\Delta = 4\pi a_2/M$ and $D_0 \equiv -3hg^2/\Delta^2$.

The (bare) dimeron propagator is a constant i/Δ and the particle propagator is given by the usual non-relativistic expression $i/(p^0 - p^2/2M)$. The dressing of the dimeron propagator is given in FIG.1 (a). Summing loop-diagrams, subtracting divergent integral at $p^0 = \vec{p}^2 = 0$ and removing the cut-off one gets the following dressed dimeron propagator:

$$iS(p) = \frac{1}{-\Delta^R + \frac{Mg^2}{4\pi}\sqrt{-Mp^0 + \frac{\vec{p}^2}{4} - i\epsilon} + i\epsilon}. \tag{3}$$

Where Δ^R is the renormalised parameter (Δ has absorbed the linear divergence).

Standard power counting shows that diagrams which contribute to leading order calculations of particle - two-body bound state scattering are those illustrated in FIG.1 (b). The sum of all these diagrams satisfies the equation represented by the second equality in FIG.1 (b) [6], [7–9]:

$$a(p,k) = K(p,k) + \frac{2\lambda}{\pi}\int_0^\infty dq\; K(p,q)\frac{q^2}{q^2 - k^2 - i\epsilon}a(q,k), \tag{4}$$

where k (p) is the incoming (outgoing) momentum, $ME = 3k^2/4 - 1/a_2^2$ is the total energy, $a(p = k, k)$ is the scattering amplitude, a_2 is the two-particle scattering length, and

$$K(p,q) = \frac{4}{3}\left(\frac{1}{a_2} + \sqrt{\frac{3}{4}p^2 - ME}\right)\frac{1}{pq}\ln\left(\frac{q^2 + pq + p^2 - ME}{q^2 - qp + p^2 - ME}\right) \tag{5}$$

Eq. (4) was first derived by Skorniakov and Ter-Martirosian (S-TM) [6] and has $\lambda = 1$ for the boson case. Three nucleons in the spin $J = 1/2$ channel obey a pair of integral equations with similar properties to this bosonic equation.

It was shown in [10] that for $\lambda = 1$ the homogeneous equation corresponding to Eq. (4) has a solution for arbitrary E. This solution is well-defined except for a normalisation constant and hence the solution of Eq. (4) contains an arbitrary parameter. The sum of

the diagrams in FIG.1 (b) is only one of the solutions. Hence, given the general solution of Eq. (4), to find this sum one would have to fix the value of the arbitrary parameter appropriately.

The fact that the homogeneous equation corresponding to Eq. (4) has a solution for arbitrary E is not surprising: since Eq. (4) corresponds to a coordinate space δ-function potential, the use of the Thomas theorem [11] combined with the Efimov effect [12] explains the existence of solutions for arbitrary E. Note that two-body forces are not actually of zero range in EFT. Although Eq. (4) can be derived from the leading order Lagrangian of EFT, this equation is not a leading order approximation of a more general equation: there are no consistent equations for renormalised amplitudes in EFT if the cut-off is removed after renormalization. The problem is that EFT is a non-renormalizable theory in the traditional sense and hence to remove all divergences which occur in the equations for amplitudes one would need to include contributions of an infinite number of counter-terms at any finite order (except perhaps leading order) approximation. Hence EFT with removed cut-off describes the amplitude for a particle scattering off a two-body bound state as a sum of an infinite number of diagrams. The EFT approach is concerned with Eq. (4) only because one of its solutions corresponds to this sum of diagrams.

A great advantage of cut-off theory is that one can write down consistent equations, and the solutions of these equations are equivalent to the renormalised (with removed cut-off) amplitudes up to the order one is working with. If working with equations of cut-off theory it is necessary to keep the cut-off finite even though at leading order the cut-off can be removed, giving Eq. (4). As the equations with finite cut-off do not correspond to any system with local (δ-function type) potential, there are no three-body bound states with arbitrarily large negative energies. The solution of the homogeneous equation corresponding to equation (4), which exists for any value of the energy, does not carry any physical information. The existence of this solution is a result of the incorrect procedure of removing the cut-off in the leading order equations of the cut-off theory. Note that the amplitude determined from the equation of cut-off theory can contain some non-perturbative contributions in addition to the sum of the infinite number of diagrams drawn in FIG.1 (b) but these non-perturbative effects can not have anything to do with non-physical solutions of the homogeneous equation.

One can still use Eq. (4) to find the amplitude for a particle scattering off two-body bound state, but one should keep in mind that it contains non-physical information encoded in the solution of the corresponding homogeneous equation.

As will be seen below the EFT approach fixes uniquely the arbitrary parameter present in the general solution of Eq. (4). This particular solution with an appropriately fixed value of the arbitrary parameter is the scattering amplitude.

One can study the asymptotic behaviour of $a(p,k)$ for large p. Up to terms decreasing as p^{-1} the function $a(p,k)$ has the form [10]:

$$a(p,k) \sim \sum_i A_i\,(k)\,p^{s_i} \tag{6}$$

where s_i are roots of the following equation:

$$1 - \frac{8\lambda}{\sqrt{3}}\frac{\sin\frac{\pi s}{6}}{s\cos\frac{\pi s}{2}} = 0. \tag{7}$$

The summation in Eq. (6) goes over all solutions of Eq. (7) for which $|\mathrm{Re}s| < 1$. Eq. (7) has two roots for which $|\mathrm{Re}s| < 1$: $s = \pm is_0$, $s_0 \approx 1$. Hence, Eq.(6) becomes:

$$a(p,k) \sim A_1\,(k)\,p^{is_0} + A_2\,(k)\,p^{-is_0} \tag{8}$$

One of the arbitrary constants $A_1\,(k)$ and $A_2\,(k)$ is determined by the other when this solution is joined to the solution in the region of small p. Hence the solution of Eq. (4) depends on a single arbitrary parameter. The asymptotic behaviour of the solution of the homogeneous equation corresponding to Eq. (4) is evidently the same.

Iterating the equation (4) one gets a series which is equivalent to the sum of the diagrams in FIG.1 (b). As s_0 does not have an expansion in λ, it should be clear that for the sum of the considered diagrams (if it exists) the parameters $A_1\,(k)$ and $A_2\,(k)$ must be vanishing.

Hence the EFT with removed cut-off supports the conclusion drawn from general considerations, namely that the non-physical solution of the homogeneous equation has to be eliminated.

To find the sum of the considered infinite number of diagrams one needs to construct a solution with non-oscillating asymptotic behaviour i.e. with vanishing $A_1(k)$ and $A_2(k)$. Note that there is only one solution with such asymptotic behaviour.

To summarise, the leading order EFT for a spinless particle scattering off a two-body bound state leads to the equation of S-TM together with a boundary condition at the origin (in configuration space) which eliminates the oscillating behaviour. Hence EFT resolves quite naturally the problem of the choice for the arbitrary parameter present in the general solution of this equation.

REFERENCES

1. U. van Kolck, nucl-th/9902015;
2. P.F. Bedaque, H.-W. Hammer, and U. van Kolck, *Phys.Rev.Lett.* **82** (1999) 463, `nucl-th/9809025`;
3. P.F. Bedaque, H.-W. Hammer, and U. van Kolck, nucl-th/9811046;
4. D.B. Kaplan, *Nucl. Phys.* **B494** (1997) 471;
5. U. van Kolck, in *Proceedings of the Workshop on Chiral Dynamics 1997, Theory and Experiment*, A. Bernstein, D. Drechsel, and T. Walcher (editors), Springer–Verlag, 1998, `hep-ph/9711222`; in Ref. [13]; `nucl-th/9808007`;
6. G.V. Skorniakov and K.A. Ter-Martirosian, *Sov. Phys. JETP* **4** (1957) 648;
7. P.F. Bedaque, in Ref. [13], `nucl-th/9806041`;
8. P.F. Bedaque and U. van Kolck, *Phys. Lett.* **B428** (1998) 221;
9. P.F. Bedaque, H.-W. Hammer, and U. van Kolck, *Phys. Rev.* **C58** (1998) R641;
10. G.S. Danilov, *Sov. Phys. JETP* **13** (1961) 349;
11. L.H. Thomas, *Phys. Rev.* **47** (1935) 903;
12. V.N. Efimov, *Sov. J. Nucl. Phys.* **12** (1971) 589;
13. "Nuclear Physics with Effective Field Theory", R. Seki, U. van Kolck, and M.J. Savage (editors), World Scientific, 1998.